INVARIANT DENSITIES AND MEAN ERGODICITY OF MARKOV OPERATORS

BY

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ABSTRACT

We prove that a Markov operator T on L_1 has an invariant density if and only if there exists a density f that satisfies $\limsup_{n\to\infty} \|T^n f - f\| < 2$. Using this result, we show that a Frobenius–Perron operator P is mean ergodic if and only if there exists a density w such that $\limsup_{n\to\infty} \|P^n f - w\| < 2$ for every density f. Corresponding results hold for strongly continuous semigroups.

1. Notations

Let (Ω, Σ, μ) be a σ-finite measure space. Given a set $X \in \Sigma$, we will denote by χ_X the function which is equal to 1 on X and 0 on $\Omega - X$. By $\mathcal{D}$ denote the set of all **densities** on Ω, i.e.,

$$\mathcal{D} = \{f \in L_1(\Omega) : f \geq 0, \|f\| = 1\},$$

where $\|\cdot\|$ is the norm in $L_1(\Omega)$. A linear operator $T\colon L_1(\Omega) \to L_1(\Omega)$ is called the **Markov operator** if $T(\mathcal{D}) \subseteq \mathcal{D}$. Let us fix a notation

$$\mathcal{T} := (T_t)_{t\in J}, \text{ where } J = \mathbb{N} \text{ or } J = \mathbb{R}_+,$$

for a one-parameter **Markov semigroup**, which means that $\mathcal{T}$ consists of Markov operators. In the case of $J = \mathbb{N}$, we will say that $\mathcal{T}$ is **discrete**, and in

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the case of $J = \mathbb{R}_+$ we always assume that $\mathcal{T}$ is **strongly continuous**. Let us denote by

$$\mathcal{A}_\tau(\mathcal{T}) := \frac{1}{\tau}\sum_{k=0}^{\tau-1} T^k \quad (\text{whenever } \tau \in J := \mathbb{N})$$

and

$$\mathcal{A}_\tau(\mathcal{T}) := \frac{1}{\tau}\int_0^\tau T_t dt \quad (\text{whenever } \tau \in J := \mathbb{R}_+)$$

the **Cesàro means** of the semigroup $\mathcal{T}$, and recall that $\mathcal{T}$ is **mean ergodic** if the norm limit $\lim_{t\to\infty} \mathcal{A}_t(\mathcal{T})f$ exists for all $f \in L_1(\Omega)$. It is well known that $\|\cdot\|$-$\lim_{t\to\infty} \mathcal{A}_t(\mathcal{T})f$ exists for an $f \in L_1(\Omega)$, whenever the set $\{\mathcal{A}_t(\mathcal{T})f\}_{t\in J}$ is conditionally weakly compact. $\mathcal{T}$ is called **weakly almost periodic** whenever the orbit $\{T_t f\}_{t\in J}$ is conditionally weakly compact for each $f \in L_1(\Omega)$. Clearly, any weakly almost periodic semigroup is mean ergodic. A density f is called **$\mathcal{T}$-invariant** if $T_t f = f$ for all $t \in J$. Of course, $f \in \mathcal{D}$ is $\mathcal{T}$-invariant for a discrete semigroup $\mathcal{T} = (T^n)_{n=1}^\infty$ if and only if it is **T-invariant** for a single operator T, i.e., $Tf = f$. It follows easily from weak compactness of any set of the kind

$$\{0 \le f \le u : f \in L_1(\Omega)\} \quad (u \in L_1(\Omega))$$

that a Markov semigroup $\mathcal{T}$ is weakly almost periodic, provided the condition that $\mathcal{T}$ possesses an invariant density u which satisfies $u(x) > 0$ a.e. on Ω.

2. Existence of an invariant density

The problem of existence of invariant densities is one of the central problems in the theory of Markov operators. There are many results in this direction [HK], [Su], [Kr, S3.4], [So], etc. Here we present a criterion for the existence of an invariant density which seems to be rather simple for applications. It will be used to obtain a condition for the mean ergodicity of the Frobenius–Perron semigroup.

THEOREM 1: *For a Markov semigroup $\mathcal{T}$, the following conditions are equivalent:*

(i) *$\mathcal{T}$ has an invariant density;*

(ii) $\limsup_{t\to\infty} \|f - T_t f\| < 2$ *for some density f;*

(iii) $\limsup_{t\to\infty} \|d - T_t g\| < 2$ *for some pair of densities d, g.*

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial, and for the proof of (iii) $\Rightarrow$ (ii) it is enough to pick $f = (d+g)/2$.

(ii) $\Rightarrow$ (i): We remark that from the equality $\|f-g\| = \|f\| + \|g\| - 2\|f \wedge g\|$, which holds for all $f, g \in L_1^+(\Omega)$, it follows that (ii) is equivalent to

$$(1) \qquad (\exists f \in \mathcal{D}) \quad \liminf_{t\to\infty} \|f \wedge T_t f\| > 0.$$

Thus we may assume that condition (1) holds for a density f.

(I) First we consider the discrete case $\mathcal{T} = (T^n)_{n=1}^{\infty}$. Without any loss of generality we may assume that $f \in L_\infty(\Omega)$. Applying [Kr, Thm. 3.4.6] to T, we obtain the decomposition of Ω into two disjoint sets C, D that satisfy the following properties:

(*) there exists a $p \in L_1^+(\Omega)$ with $Tp = p$ and $C = \{p > 0\}$, and

(**) there exists a weakly wandering $h \in L_\infty^+(\Omega)$ with $D = \{h > 0\}$.

It is enough to show that $p \neq 0$. Let $\|\sum_{\nu=1}^{\infty} T^{*k_\nu} h\|_\infty < \infty$ for some strictly increasing sequence k_ν of naturals (such a sequence exists in view of the weakly wandering of h). Given $\varepsilon > 0$, set $A_\varepsilon := \{h \geq \varepsilon\}$; then

$$\begin{aligned}\sum_{\nu=1}^{\infty} \|\chi_{A_\varepsilon} \cdot (f \wedge T^{k_\nu} f)\| &\leq \sum_{\nu=1}^{\infty} \|\chi_{A_\varepsilon} \cdot T^{k_\nu} f\| \\ &= \int_{A_\varepsilon} \sum_{\nu=1}^{\infty} T^{k_\nu} f \, d\mu \\ &\leq \varepsilon^{-1} \int_\Omega h \cdot \sum_{\nu=1}^{\infty} T^{k_\nu} f \, d\mu \\ &= \varepsilon^{-1} \Big(\sum_{\nu=1}^{\infty} T^{*k_\nu} h \Big)(f) \\ &\leq \varepsilon^{-1} \cdot \|f\| \cdot \Big\| \sum_{\nu=1}^{\infty} T^{*k_\nu} h \Big\|_\infty < \infty,\end{aligned}$$

and consequently, $\lim_{\nu\to\infty} \|\chi_{A_\varepsilon} \cdot (f \wedge T^{k_\nu} f)\| = 0$ for all $\varepsilon > 0$. Now in view of $A_\varepsilon \uparrow D$ $(\varepsilon \downarrow 0)$, we obtain that $\lim_{\nu\to\infty} \|\chi_D \cdot (f \wedge T^{k_\nu} f)\| = 0$, and

$$\begin{aligned}\limsup_{\nu\to\infty} \int_C f \wedge T^{k_\nu} f \, d\mu &\geq \limsup_{\nu\to\infty} \int_\Omega f \wedge T^{k_\nu} f \, d\mu - \lim_{\nu\to\infty} \|\chi_D \cdot (f \wedge T^{k_\nu} f)\| \\ &\geq \liminf_{n\to\infty} \|f \wedge T^n f\| > 0.\end{aligned}$$

In particular, the set C has a positive measure. Thus $p \neq 0$ and $w = \|p\|^{-1} p$ is an invariant density of T.

(II) Now let $\mathcal{T} = (T_t)_{t\geq 0}$ be a strongly continuous Markov semigroup. Set $T = T_1$; then condition (1) implies that

$$\liminf_{n\to\infty} \|f \wedge T^n f\| > 0,$$

and from part (I) of the proof it follows that there exists a density u_1 such that $Tu_1 = u_1$. Clearly

$$u := \int_0^1 T_t u_1 dt$$

is a $\mathcal{T}$-invariant density, since

$$u = \int_0^1 T_t u_1 dt = \int_s^{1+s} T_t u_1 dt = T_s u (\forall s \geq 0). \qquad \blacksquare$$

3. Weak almost periodicity of bounded semigroups on L_1

The rather surprising fact, that any mean ergodic Markov semigroups is weakly almost periodic, was obtained recently by Komornik [Ko, Prop. 1.4(i)] and independently by Kornfeld and Lin [KL, Thm. 1.2] for discrete semigroups. We extend the result to bounded semigroups, with a different idea for the proof.

THEOREM 2: *Any mean ergodic bounded semigroup of positive operators $\mathcal{T}$ on L_1 is weakly almost periodic.*

In the proof below, we will use the following simple inequality:

$$\inf\{\|\xi\| : \xi \in \Xi\} \leq \|f\| \quad (\forall f \in \overline{\mathrm{co}}(\Xi)), \tag{2}$$

which is true for any $\Xi \subseteq L_1^+(\Omega)$, due to additivity of the norm on $L_1^+(\Omega)$.

Proof: It is enough to show that the orbit $\{T_t f\}_{t \in J}$ is conditionally weakly compact for all $f \in L_1^+(\Omega)$.

Fix an $f \in L_1^+(\Omega)$ and let $u := \lim_{t\to\infty} \mathcal{A}_t(\mathcal{T})f$. If $u = 0$, set $Q = I := Id|_{L_1(\Omega)}$. If $u \neq 0$, take the u-support projection $P = P_u$:

$$P_u g = \chi_{\{u>0\}} \cdot g \quad (\forall g \in L_1(\Omega)).$$

It is clear that P satisfies $0 \leq P \leq I$, and $T_t P = P T_t P$ since $L_1(\{u > 0\})$ is $\mathcal{T}$-invariant. Set $Q = I - P$ and notice that $QT_t = QT_tQ$ for all $t \in J$.

Since $\lim_{t\to\infty} \|u - \mathcal{A}_t(\mathcal{T})f\| = 0$ and $Qu = 0$, we have $\lim_{t\to\infty} \|Q\mathcal{A}_t(\mathcal{T})f\| = 0$. Applying the inequality (2) to the set $\Xi := \{QT_t f\}_{t\in J}$, we obtain that

$$\|QT_{n_i} f\| \to 0 \quad (i \to \infty)$$

for some increasing sequence (n_i), and consequently

$$\begin{aligned}\limsup_{t\to\infty} \|QT_t f\| &= \limsup_{t\to\infty} \|QT_t T_{n_i} f\| \\ &= \limsup_{t\to\infty} \|QT_t Q T_{n_i} f\| \\ &\leq \sup_{t\in J} \|T_t\| \cdot \|QT_{n_i} f\| \to 0 \quad (i \to \infty).\end{aligned}$$

Thus $\lim_{t\to\infty} \|QT_t f\| = 0$. In the case $u = 0$, the proof is finished already, since $Q = I$. Let $u \neq 0$. The set $\bigcup_{l=1}^{\infty}[-lu, lu]$ is norm-dense in $L_1(\{u > 0\}) = P(L_1(\Omega))$. By invariance of u

$$T_t([-lu, lu]) \subseteq [-lu, lu] \quad (\forall t \in J),$$

and since $\mathcal{T}$ is bounded, for each $\varepsilon > 0$, there exists l_ε such that

$$\limsup_{t\to\infty} \operatorname{dist}(T_t f, [-lu, lu]) = \limsup_{t\to\infty} \operatorname{dist}(PT_t f, [-lu, lu]) \leq \varepsilon$$

for all $l \geq l_\varepsilon$. Henceforth $\{T_t f\}_{t\in J}$ is conditionally weakly compact, since $[-lu, lu]$ is weakly compact and since $\varepsilon > 0$ was chosen arbitrary. ∎

4. Weak almost periodicity for Frobenius–Perron semigroups

We start with the following definitions. A transformation $\tau\colon \Omega \to \Omega$ is called **measurable** if $\tau^{-1}(A) \in \Sigma$ for all $A \in \Sigma$. A measurable transformation $\tau\colon \Omega \to \Omega$ is called **nonsingular** if $\mu(\tau^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. It follows from the Radon–Nikodim theorem that for any nonsingular transformation τ the equality

$$\int_A Pf\,d\mu = \int_{\tau^{-1}(A)} f\,d\mu \quad (A \in \Sigma)$$

defines a unique operator P on $L_1(\Omega)$, which is called the **Frobenius–Perron operator** corresponding to τ. It is easy to see that any Frobenius–Perron operator is a Markov operator. When a semigroup $(\tau_t)_{t\in J}$ of nonsingular transformations on (Ω, Σ, μ) is given, $(P_{\tau_t})_{t\in J}$ is called a **Frobenius–Perron semigroup.**

Let now $\mathcal{P} = (P_{\tau_t})_{t\in J}$ be a discrete or a strongly continuous Frobenius–Perron semigroup, associated with a semigroup $(\tau_t)_{t\in J}$ of nonsingular transformations $\tau_t\colon \Omega \to \Omega$. We apply Theorem 1 to obtain the following criteria of the mean ergodicity of $\mathcal{P}$.

THEOREM 3: *For a Frobenius–Perron semigroup $\mathcal{P}$ the following conditions are equivalent:*

(i) *$\mathcal{P}$ is weakly almost periodic;*

(ii) *$\mathcal{P}$ is mean ergodic;*

(iii) *there exists a density w such that $\limsup_{t\to\infty} \|P_{\tau_t} f - w\| < 2$ for every density f.*

Proof: The equivalence (i) ⇔ (ii) follows from Theorem 2.

(ii) ⇒ (iii): Take a density d such that $d(x) > 0$ a.e. on Ω; then the density $w = \lim_{t\to\infty} \mathcal{A}_t(\mathcal{P})d$ satisfies (iii). Indeed, let $f \in \mathcal{D}$; then

$$\inf_t \|\mathcal{A}_t(\mathcal{P})f - w\| \leq \lim_{t\to\infty} \|\mathcal{A}_t(\mathcal{P})(f-d)\| \leq \|f-d\| < 2.$$

Since $\mathcal{A}_t(\mathcal{P})f \in \overline{\mathrm{co}}\{P_{\tau_t}f : t \in J\}$, the inequality above shows that there exists an element $a \in \mathrm{co}\{P_{\tau_t}f : t \in J\}$ with $\|a - w\| < 2$ and hence $\|P_{\tau_{t_0}} - w\| < 2$ for some $t_0 \in J$. But then

$$\limsup_{t\to\infty} \|P_{\tau_t}f - w\| = \limsup_{s\to\infty} \|P_{\tau_s}(P_{\tau_{t_0}}f - w)\| \leq \|P_{\tau_{t_0}} - w\| < 2,$$

which is what is required.

(iii) ⇒ (ii): Denote by $\mathcal{D}_{\mathcal{P}}$ the set of all $\mathcal{P}$-invariant densities. Take a finite measure μ_1 which is equivalent to the initial σ-finite measure μ on Ω. $\mathcal{D}_{\mathcal{P}}$ is not empty by Theorem 1, so we can define

$$\alpha := \sup\{\mu_1(E) : E = \{d > 0\} \text{ for some } d \in \mathcal{D}_p\},$$

and $0 < \alpha < \infty$. Let $(d_n)_{n=1}^{\infty}$ be in $\mathcal{D}_p$ with $\mu_1(\{d_n > 0\}) \to \alpha$. Put $a = \sum_{n=1}^{\infty} 2^{-n} d_n$, and denote $A = \{a > 0\}$. Then $a \in \mathcal{D}_p$ and $\mu_1(A) = \alpha$, i.e., a is an invariant density with maximal support. Let $A_1 = \bigcup_{t\in J} \tau_t^{-1}(A)$. Obviously $\tau_t(A_1) \subseteq A_1$ for any $t \in J$. But $B := \Omega - A_1 = \{x : \tau_t x \notin A \forall t \in J\}$ is also obviously invariant. Hence $L_1(B)$ is invariant for all P_{τ_t}. By (iii), $\chi_B \neq 0$, so Theorem 1 yields a density supported in B invariant for all P_{τ_t}, which contradicts maximality of A. Hence $\Omega = \bigcup_{t\in J} \tau_t^{-1}(A)$.

Since, obviously, $\tau_t(A) \subset A$ for all $t \in J$, we obtain

$$\lim_{t\to\infty} \int_{\Omega - A} P_{\tau_t} f d\mu = \lim_{t\to\infty} \int_{\Omega - \tau_t^{-1}(A)} f d\mu = 0 \quad (\forall f \in \mathcal{D}). \tag{3}$$

On the other hand, the restriction $\mathcal{P}|_{L_1(A)}$ of $\mathcal{P}$ on $L_1(A)$ is mean ergodic, since the semigroup $\mathcal{P}|_{L_1(A)}$ has the almost everywhere positive (on the set A) invariant density a. Consequently, (3) implies that $\mathcal{P}$ is mean ergodic. ∎

We remark that the implication (i) ⇔ (ii) ⇒ (iii) is true for any Markov semigroup. The following example of Komornik [Ko, Example 4.1] shows that the implication (iii) ⇒ (ii) does not hold in general.

Example: Let $\Omega = \mathbb{N}$, and let Σ be the algebra of all subsets of $\mathbb{N}$ and μ be the counting measure. Thus $L_1(\Omega) = \ell^1$. Define an operator T on ℓ^1 as follows:

$$Te_{k+1} := 2^{-k} \cdot e_1 + (1 - 2^{-k}) \cdot e_{k+2} \quad (\forall k \geq 0).$$

So the defined Markov semigroup $\mathcal{T} = (T^n)_{n=1}^{\infty}$ obviously satisfies the condition (iii) of Theorem 3 with $w = e_1$. But it is easy to see that for any density d the sequence $(\frac{1}{n}\sum_{i=0}^{n-1} T^i d)_{n=1}^{\infty}$ converges only if d is equal to e_1. In particular, T is not mean ergodic.

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References

[HK] A. B. Hajian and S. Kakutani, *Weakly wandering sets and invariant measures*, Transactions of the American Mathematical Society **110** (1964), 136–151.

[Ko] J. Komornik, *Asymptotic periodicity of Markov and related operators*, Dynamics Reported, Expositions on Dynamical Systems (New Series) **2**, Springer, Berlin, 1993, pp. 31–68.

[KL] I. Kornfeld and M. Lin, *Weak almost periodicity of L_1-contractions and coboundaries of non-singular transformations*, Studia Mathematica **138** (2000), 225–240.

[Kr] U. Krengel, *Ergodic Theorems*, De Gruyter, Berlin–New York, 1985.

[So] J. Socala, *On the existence of invariant densities for Markov operators*, Annales Polonici Mathematici **48** (1988), 51–56.

[Su] L. Sucheston, *On existence of finite invariant measures*, Mathematische Zeitschrift **86** (1964), 327–336.